

## A DETERMINANT FORMULA FOR THE RELATIVE CONGRUENT ZETA FUNCTION IN CYCLOTOMIC FUNCTION FIELDS

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ABSTRACT. In this paper we give a determinant formula for the relative congruent zeta function in cyclotomic function fields.

### 1. Introduction

Let  $k = \mathbb{F}_q(T)$  be the rational function field over the finite field  $\mathbb{F}_q$  and  $\mathbb{A} = \mathbb{F}_q[T]$ . Let  $\infty = (1/T)$  be the prime of  $k$  associated to  $1/T$ , which is called the infinite prime of  $k$ . Write  $\mathbb{A}^+ = \{1 \neq M \in \mathbb{A} : M \text{ is monic}\}$  and  $\mathbb{A}_{\text{irr}}^+ = \{P \in \mathbb{A}^+ : P \text{ is irreducible}\}$  for simplicity. For any  $M \in \mathbb{A}^+$ , write  $K_M$  for the  $M$ th cyclotomic function field and  $K_M^+$  for the maximal real subfield of  $K_M$ .

It is known that there exists a polynomial  $P_{K_M}(X) \in \mathbb{Z}[X]$  such that

$$\zeta(s, K_M) = \frac{P_{K_M}(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})},$$

where  $\zeta(s, K_M)$  is the congruence zeta function of  $K_M$ , and  $P_{K_M}(1)$  is equal to the divisor class number  $h_{K_M}$  of  $K_M$ . Let  $\zeta^{(-)}(s, K_M) = \zeta(s, K_M)/\zeta(s, K_M^+)$ , called the relative congruence zeta function of  $K_M$  and  $P_{K_M}^{(-)}(X) = P_{K_M}(X)/P_{K_M^+}(X)$ , where  $P_{K_M^+}(X)$  is the polynomial corresponding to the congruence zeta function  $\zeta(s, K_M^+)$  for  $K_M^+$ . Then we have  $\zeta^{(-)}(s, K_M) = P_{K_M}^{(-)}(q^{-s})$  and  $P_{K_M}^{(-)}(1) = h_{K_M}^-$ , where  $h_{K_M}^- = h_{K_M}/h_{K_M^+}$  is the relative divisor class number of  $K_M$ .

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In recent paper [5], Shiomi has expressed the polynomial  $P_{K_M}^{(-)}(X)$  as determinant of matrix  $D_M^{(-)}(X)$  up to some polynomial  $J_{K_M}^{(-)}(X)$ . Since  $h_{K_M}^- = P_{K_M}^{(-)}(1)$ , this determinant formula for  $P_{K_M}^{(-)}(X)$  can be regarded as generalization of that for relative class number  $h_{K_M}^{(-)}$  ([4], [2], [1]).

The aim of this paper is to give another determinant formula for the polynomial  $P_{K_M}^{(-)}(X)$ . In contrast to the fact that in the determinant in [5], the matrix  $D_M^{(-)}(X)$  is a product of matrices with entries containing polynomials with complex coefficients, the entries of matrices in our determinant formula are polynomials with integer coefficients.

## 2. Zeta and $L$ -functions

Let  $F$  be a finite extension of  $k$  which is contained in some cyclotomic extension  $K_M$ . Let  $N \in \mathbb{A}^+$  be the conductor of  $F$ , that is,  $K_N$  is the smallest cyclotomic function field containing  $F$ . Let  $\zeta(s, F)$  be the congruence zeta function of  $F$  given by

$$\zeta(s, F) = \prod_{\mathfrak{p}} \left(1 - \frac{1}{N\mathfrak{p}^s}\right)^{-1},$$

where  $\mathfrak{p}$  runs over all primes of  $F$ . It is well known that there exists a polynomial  $P_F(X) \in \mathbb{Z}[X]$  of degree  $2g$ , where  $g$  is the genus of  $F$ , such that

$$(2.1) \quad \zeta(s, F) = \frac{P_F(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})}.$$

Moreover, the polynomial  $P_F(X)$  satisfies  $P_F(0) = 1$  and  $P_F(1) = h_F$ , where  $h_F$  is the divisor class number of  $F$ . Let  $\mathcal{O}_F$  be the integral closure of  $\mathbb{A}$  in  $F$  and  $\zeta(s, \mathcal{O}_F)$  be the zeta function of  $\mathcal{O}_F$  given by

$$\zeta(s, \mathcal{O}_F) = \prod_{\mathfrak{p}} \left(1 - \frac{1}{N\mathfrak{p}^s}\right)^{-1},$$

where  $\mathfrak{p}$  runs over all prime ideals of  $\mathcal{O}_F$ . Then the functions  $\zeta(s, F)$  and  $\zeta(s, \mathcal{O}_F)$  satisfy the following equality

$$(2.2) \quad \zeta(s, F) = \zeta(s, \mathcal{O}_F)(1 - q^{-s})^{-[F^+ : k]}.$$

Let  $X_F$  be the group of primitive Dirichlet characters of  $\mathbb{A}$  associated to  $F$ . For  $\chi \in X_F$ , let  $L(s, \chi)$  be the  $L$ -function associated to  $\chi$  given by

$$L(s, \chi) = \prod_{P \in \mathbb{A}_{\text{irr}}^+} (1 - \chi(P)q^{-s \deg P})^{-1}.$$

Then we have

$$(2.3) \quad \zeta(s, \mathcal{O}_F) = \prod_{\chi \in X_F} L(s, \chi).$$

Let  $\chi_0 \in X_F$  denote the trivial character. Since  $L(s, \chi_0) = (1 - q^{1-s})^{-1}$ , from (2.1), (2.2) and (2.3), we get

$$(2.4) \quad \prod_{\chi_0 \neq \chi \in X_F} L(s, \chi) = (1 - q^{-s})^{[F^+ : k] - 1} P_F(q^{-s}).$$

For any  $\chi \in X_F$ , let  $F_\chi \in \mathbb{A}^+$  be the conductor of  $\chi$  and  $\tilde{\chi} = \chi \circ \pi_\chi$ , where  $\pi_\chi : (\mathbb{A}/N\mathbb{A})^* \rightarrow (\mathbb{A}/F_\chi\mathbb{A})^*$  is the canonical homomorphism. Then we have

$$(2.5) \quad L(s, \tilde{\chi}) = L(s, \chi) \prod_{Q \in \mathbb{A}_{\text{irr}}^+, Q|N} (1 - \chi(Q)q^{-s \deg Q}).$$

Thus, by (2.4) and (2.5), we have

$$(2.6) \quad \prod_{\chi_0 \neq \chi \in X_F} L(s, \tilde{\chi}) = (1 - q^{-s})^{[F^+ : k] - 1} P_F(q^{-s}) J_F(q^{-s}),$$

where  $J_F(X)$  is the polynomial given by

$$J_F(X) = \prod_{\chi_0 \neq \chi \in X_F} \prod_{Q \in \mathbb{A}_{\text{irr}}^+, Q|N} (1 - \chi(Q)X^{\deg Q}).$$

Let  $\zeta^{(-)}(s, F) = \zeta(s, F)/\zeta(s, F^+)$  be the relative congruence zeta function of  $F$  and  $P_F^{(-)}(X) = P_F(X)/P_{F^+}(X)$ . Then by (2.1) and (2.4), we have

$$\zeta^{(-)}(s, F) = P_F^{(-)}(q^{-s}) = \prod_{\chi \in X_F^-} L(s, \chi)$$

where  $X_F^- = X_F \setminus X_{F^+}$ . Finally, by (2.5), we have

$$(2.7) \quad \prod_{\chi \in X_F^-} L(s, \tilde{\chi}) = P_F^{(-)}(q^{-s}) J_F^{(-)}(q^{-s}),$$

where

$$J_F^{(-)}(X) = \prod_{\chi \in X_F^-} \prod_{Q \in \mathbb{A}_{\text{irr}}^+, Q|N} (1 - \chi(Q)X^{\deg Q}).$$

Finally we give some remarks on the polynomial  $J_F^{(-)}(X)$ . They satisfy the following equality ([5, Proposition 3.1])

$$J_F^{(-)}(X) = \prod_{Q \in \mathbb{A}_{\text{irr}}^+, Q|N} \frac{(1 - X^{f_Q \deg Q})^{g_Q}}{(1 - X^{f_Q^+ \deg Q})^{g_Q^+}},$$

where  $f_Q, f_Q^+$  are the residue class degrees of  $Q$  in  $F/k, F^+/k$ , respectively and  $g_Q, g_Q^+$  are the number of primes over  $Q$  in  $F, F^+$ , respectively. Hence we see that  $J_F^{(-)}(X) \in \mathbb{Z}[X]$  and in particular,  $J_F^{(-)}(X) = 1$  if  $N$  is a power of some  $Q \in \mathbb{A}_{\text{irr}}^+$ .

### 3. A determinants formula for $P_{K_M}^{(-)}(X)$

From now on, we consider the case  $F = K_M$ , the  $M$ th cyclotomic function field for  $M \in \mathbb{A}^+$ . Let  $\mathcal{R}_M = (\mathbb{A}/M\mathbb{A})^*/\mathbb{F}_q^*$ . For  $\alpha \in (\mathbb{A}/M\mathbb{A})^*$ , there exists a unique polynomial  $A_\alpha \in \mathbb{A}$  such that  $\deg A_\alpha < \deg M$  and  $A_\alpha + M\mathbb{A} = \alpha$ . Write  $\text{sgn}_M(\alpha) \in \mathbb{F}_q^*$  for the leading coefficient of  $A_\alpha$ ,  $\deg_M(\alpha) := \deg A_\alpha$ . For  $\sigma \in \text{Gal}(K_M/k)$ , let  $\alpha_\sigma$  be the element of  $(\mathbb{A}/M\mathbb{A})^*$ , which corresponds to  $\sigma$  under the canonical isomorphism  $\text{Gal}(K_M/k) \cong (\mathbb{A}/M\mathbb{A})^*$ . We define a function  $f : \text{Gal}(K_M/k) \rightarrow \mathbb{Z}[X]$  by

$$f(\sigma) := \begin{cases} X^{\deg_M(\alpha_\sigma)}, & \text{if } \text{sgn}_M(\alpha_\sigma) = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Now we define the matrix

$$E_{K_M}^{(-)}(X) = (f(\sigma\tau^{-1}) - f(\sigma(\tau/\text{sgn}_M(\alpha_\tau))^{-1}))_{\sigma,\tau},$$

where  $\sigma, \tau \in \text{Gal}(K_M/k)$  with  $\text{sgn}_M(\alpha_\sigma) \neq 1, \text{sgn}_M(\alpha_\tau) \neq 1$ . Then we have the following main theorem, which gives a determinant formula for the relative congruence zeta functions in cyclotomic function fields.

**THEOREM 3.1.** *With notations as above, we have*

$$\det E_{K_M}^{(-)}(X) = P_{K_M}^{(-)}(X) J_{K_M}^{(-)}(X).$$

*Proof.* For  $\chi \in X_{K_M}^-$ , as in the proof of [5, Theorem 3.1] or [3, Lemma 3], we have

$$\begin{aligned} L(s, \tilde{\chi}) &= \sum_{\alpha \in (\mathbb{A}/M\mathbb{A})^*, \text{sgn}_M(\alpha)=1} \tilde{\chi}(\alpha) q^{-\deg_M(\alpha)s} \\ &= \sum_{\sigma \in \text{Gal}(K_M/\mathbb{k})} \tilde{\chi}(\alpha_\sigma) f(\sigma)(q^{-s}). \end{aligned}$$

Noting that under the isomorphism  $\text{Gal}(K_M/\mathbb{k}) \cong (\mathbb{A}/M\mathbb{A})^*$ ,  $\{\sigma \in \text{Gal}(K_M/\mathbb{k}) \mid \text{sgn}_M(\alpha_\sigma) = 1\}$  represents  $\text{Gal}(K_M/\mathbb{k}) / \text{Gal}(K_M^+/\mathbb{k})$ , we have, from [1, Corollary 2.2(ii)],

$$\prod_{\chi \in X_{K_M}^-} L(s, \tilde{\chi}) = \det \left( f(\sigma\tau^{-1})(q^{-s}) - f(\sigma(\tau/\text{sgn}_M(\alpha_\tau))^{-1})(q^{-s}) \right)_{\sigma, \tau}.$$

Now from (2.7), we get the result.  $\square$

**EXAMPLE 3.2.** For  $q = 3$ ,  $M = T^2 + 1$ , since  $M$  is irreducible, we have  $\det E_{K_M}^{(-)}(X) = P_{K_M}^{(-)}(X)$ . Then we have  $\{A_{\alpha_\sigma} \mid \sigma \in \text{Gal}(K_M/\mathbb{k}), \text{sgn}_M(\alpha_\sigma) \neq 1\} = \{-1, -T, -T + 1, -T + 2\}$  and so

$$\begin{aligned} P_{K_M}^{(-)}(X) &= \det E_{K_M}^{(-)}(X) \\ &= \begin{vmatrix} 1 & -X & X & X \\ X & 1 & X & -X \\ X & X & 1 & X \\ X & -X & -X & 1 \end{vmatrix} \\ &= 1 - 2X^2 + 9X^4. \end{aligned}$$

Thus we have the same result as in [5, Example 5.1].

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